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A STATISTICAL OPTIMIZING NAVIGATION PROCEDURE FOR SPACE FLIGHT

by

Richard H. Battin

September 1961

INSTRUMENTATION LABORATORY

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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ABSTRACT

In a typical self-contained space navigation system celestial observation data are gathered and processed to produce estimated velocity corrections. The results of this paper provide a basis for determining the best celestial measurements and the proper times to implement velocity corrections.

Fundamental to the navigation system is a procedure for processing celestial measurement data which permits incorporation of each individual measurement as it is made to provide an improved estimate of position and velocity. In order to "optimize" the navigation, a statistical evaluation of a number of alternative courses of action is made. The various alternatives, which form the basis of a decision process, concern the following:

1. Which star and planet combination provide the "best" available observation ?
2. Does the best observation give a sufficient reduction in the predicted target error to warrant making the measurement?
3. Is the uncertainty in the indicated velocity correction a small enough percentage of the correction itself to justify an engine restart and propellant expenditure?

Numerical results are presented which illustrate the effectiveness of this approach to the space navigation problem.

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1. INTRODUCTION

During the past two years, the problems of guiding a space vehicle during the midcourse phase of its mission have been extensively explored at the MIT Instrumentation Laboratory. Following the specific demonstration of the technical feasibility of an unmanned photographic reconnaissance flight to the planet Mars reported by Laning, Frey, and Trageser [1], the detailed navigational aspects of such a venture were developed [2] by Dr. J. H. Laning, Jr., and the present author. Later, a variable time of arrival navigation theory was devised [3] and contrasted with the earlier fixed time of arrival scheme. More recently, the question of optimum utilization of navigation data has been given considerable study. It is the solution of this problem which forms the subject of the present paper.

The general method of navigation is based on perturbation theory so that only deviations in position and velocity from a reference path are utilized. Data is gathered by an optical angle measuring device and processed by a spacecraft digital computer. Periodically, small changes in the spacecraft velocity are implemented by a propulsion system as directed by the computer.

Basically, three problems are considered in this paper: (1) to identify the best sources of data available to the space vehicle navigator; (2) to define the optimum linear operations for processing the data in a manner consistent with the mission objectives; and (3) to minimize both the amount of navigational data and the number of corrective maneuvers required without unduly compromising mission accuracy.

The formulation of an optimum linear estimator as a recursion operation in which the current best estimate is combined with newly acquired information to produce a still better estimate was presented by Kalman [4]. The author is indebted to Dr. Stanley F. Schmidt for directing his attention to Kalman's excellent work. In fact, the original application of Kalman's theory to space navigation was made by Schmidt [5] and his associates.

The work described in the following sections of this paper was done without any detailed knowledge of Schmidt's activities. As a result of this independent approach, several new and interesting ideas have developed. Specifically, an extremely simple derivation of the optimum linear operator has been achieved using only the basic technique of least squares estimation. In addition, the following new results are noted: (1) the mean-squared velocity correction is expressed directly in terms of initial orbital injection errors, the errors associated with navigational measurements, and the errors in establishing the desired velocity correction -- hence, a statistical simulation of the navigation scheme may be made without resorting to Monte Carlo techniques; (2) a detailed procedure for incorporating cross-correlation effects of random measurement errors in determining the optimum linear operator has been developed; and (3) the mathematical problem of determining the optimum plane in which to make a star-planet angular measurement has been solved.

Throughout the paper, we shall deal exclusively with discrete information; observations or velocity corrections are made at specific points in time which are termed "decision points." The interval between decision points is not necessarily uniform and may be selected somewhat arbitrarily; e.g., the interval length required for accurate numerical integration of the trajectory equations was used in preparing the computational data presented in Section 6.

Finally, a few remarks relevant to notational conventions are appropriate. We shall deal generally with both three- and six-dimensional vectors. A column vector of any dimension is represented by a lower case underscored letter. Matrices are denoted by capital letters and can be either square or rectangular arrays. The transpose of a vector or a matrix will be denoted by a superscript T. Thus, the scalar product of two vectors $\underline{a}$ and $\underline{b}$ will be written as $\underline{a}^T\underline{b}$. In like manner a quadratic form associated with a square matrix A will be written as $\underline{x}^T A\underline{x}$. Finally, the expected value of a random vector $\underline{x}$ will be indicated by an overscore; thus, $\overline{\underline{x}}$ denotes the average value of $\underline{x}$.

2. OUTLINE OF THE NAVIGATION AND GUIDANCE PROCEDURE

2.1 A Deterministic Method

The basic process involved in determining spacecraft position by means of a celestial fix consists fundamentally of a sequence of measurements of the angles between selected pairs of celestial objects. Three independent and precise angular measurements made at a known instant of time suffice to determine uniquely the position of the vehicle. Practical constraints, however, preclude simultaneous measurements without severely complicating the instrumentation. On the other hand, if the vehicle dynamics are governed by known laws and if deviations from a pre-determined reference trajectory are kept sufficiently small to permit a linearization of the navigation problem, then the question of simultaneous measurements loses it significance.

Under the assumptions of a linearized theory, a single observation serves to fix the position of the spacecraft in one coordinate. For example, if A_n is the angle measured at time t_n and is defined by the lines-of-sight from the vehicle to a star and to a nearby celestial body, the position of the vehicle is established along a line normal to the direction toward the near body and in the plane of the measurement. It is shown in Appendix A that the deviation in position $\delta \underline{r}_n$ of the spacecraft from the reference position is related to the deviation in angular measurement δA_n by

$$\delta A_n = \underline{h}_n^T \delta \underline{r}_n \tag{2.1}$$

if the observation is made at a known instant of time t_n. The vector $\underline{h}_n$ depends upon the geometrical configuration of the relevant celestial objects at time t_n as well as the type of measurement made.

Because of the inherent dynamic coupling of position and velocity, the result at a later time t_{n+1} of a measurement made at time t_n does not lend itself to simple geometric interpretation. In order to provide a geometrical description, it is convenient to introduce the concept of a six dimensional space in which the coordinates represent the components of both position and velocity deviations of the vehicle from the reference path as functions of time.

Points in this space are defined by the six dimensional deviation vector

$$\delta \underline{x}_n = \left\| \begin{matrix} \delta \underline{r}_n \\ \delta \underline{v}_n \end{matrix} \right\| \tag{2.2}$$

where $\delta \underline{v}_n$ is the deviation in the vector velocity of the vehicle from the reference value. The vector $\delta \underline{x}_n$ defines the "state" of the vehicle dynamics at time t_n. Transition from one state to another is provided by the matrix operation

$$\Phi_{n+1,n} = \Phi(t_{n+1}, t_n)$$

which is frequently referred to as the "transition matrix". Indeed, the relationship between $\delta \underline{x}_{n+1}$ and $\delta \underline{x}_n$ is simply

$$\delta \underline{x}_{n+1} = \Phi_{n+1,n}\, \delta \underline{x}_n \tag{2.3}$$

as shown in Section 3.4.

By means of the rectangular matrix K defined by

$$K = \left\| \begin{matrix} I \\ O \end{matrix} \right\| \tag{2.4}$$

Eq. (2.1) may be written in terms of $\delta \underline{x}_n$ as

$$\delta A_n = \underline{h}_n^T K^T \delta \underline{x}_n \tag{2.5}$$

The submatrices I and O are, respectively, the three dimensional identity and zero matrices. Now, by combining Eqs. (2.3) and (2.5)

$$\delta A_n = \underline{h}_n^T K^T \Phi_{n+1,n}^{-1} \delta \underline{x}_{n+1} \tag{2.6}$$

it is clear that the effect at time t_{n+1} of an observation at time t_n is to determine the component of the six dimensional deviation vector in the direction defined by the vector $\Phi^{T-1}_{n+1,\,n} K \underline{h}_n$. Six observations made at different times would provide a set of six equations of the form of Eq. (2.6). If no two of the component directions were parallel, then the deviation vector could be obtained by inverting the six dimensional coefficient matrix.

2.2 Statistical Parameters of the Navigation Problem

Because of the presence of instrument inaccuracies additional observations may be used to reduce the errors associated with the simple deterministic process just described. By applying least square techniques to the observed data, a more accurate estimate of position and velocity is frequently possible than could be obtained from the minimum number of measurements. For this

purpose, it is necessary to know certain statistical information with respect to the instrument inaccuracies. In a linear least squares estimation procedure all statistical calculations are based on first and second order averages and no additional statistical data is needed.

At this point of the discussion it is necessary to distinguish measured values, estimated values and true values of various quantities; e.g., $\delta \tilde{A}_n$ will be the measured value of the deviation in the angle A_n from its reference value at time t_n, δA_n the true value of the deviation, and $\delta \hat{A}_n$ the estimated value. If we write

$$\delta \tilde{A}_n = \delta A_n + \alpha_n \tag{2.7}$$

then α_n will be the error in the measurement. In the subsequent analysis α_n will be regarded as a random variable with an average value $\bar{\alpha}_n$ and a variance

$$\sigma_n^2 = \overline{\alpha_n^2} - \bar{\alpha}_n^2 \tag{2.8}$$

The possibility of cross-correlation of measurement errors will not be excluded; i.e., in general, the average $\overline{\alpha_n \alpha_m}$ may be different from zero.

In Section 4 an estimation procedure is developed for determining an optimal linear estimate of $\delta \underline{x}_n$, denoted by $\delta \hat{\underline{x}}_n$. As each measurement is made, the estimate $\delta \hat{\underline{x}}_n$ is updated by a simple recursive formula and, thereby, the problem associated with inverting sixth order matrices is avoided. An integral part of the estimation technique is the correlation matrix of the errors in the estimate. If we write

$$\delta \hat{\underline{x}}_n = \delta \underline{x}_n + \underline{e}_n \tag{2.9}$$

then

$$\underline{e}_n = \left\| \begin{matrix} \underline{\epsilon}_n \\ \underline{\delta}_n \end{matrix} \right\| \tag{2.10}$$

is the six dimensional error vector and may be partitioned as shown using $\underline{\epsilon}_n$ and $\underline{\delta}_n$ to denote, respectively, the position and velocity errors. The correlation matrix is thus defined by

$$E_n = \overline{\underline{e}_n \underline{e}_n^T} = \left\| \begin{matrix} \overline{\underline{\epsilon}_n \underline{\epsilon}_n^T} & \overline{\underline{\epsilon}_n \underline{\delta}_n^T} \\ \overline{\underline{\delta}_n \underline{\epsilon}_n^T} & \overline{\underline{\delta}_n \underline{\delta}_n^T} \end{matrix} \right\| = \left\| \begin{matrix} E_n^{(1)} & E_n^{(2)} \\ E_n^{(3)} & E_n^{(4)} \end{matrix} \right\| \tag{2.11}$$

When cross-correlation of measurement errors is considered, a correlation vector

$$\underline{\phi}_{nm} = \overline{\alpha_n \underline{e}_m} \tag{2.12}$$

is needed to represent the average value of the product of a measurement error at time t_n and the estimation error at time t_m.

It is important to distinguish between a new estimate $\delta \hat{\underline{x}}_n$, obtained by incorporating an observation at time t_n, and an estimate simply extrapolated from a previous estimate. For the latter case, the notation $\delta \hat{\underline{x}}'_n$ is used where

$$\delta \hat{\underline{x}}'_n = \Phi_{n,\, n-1}\, \delta \hat{\underline{x}}_{n-1} \tag{2.13}$$

In like manner, we define an extrapolated error vector $\underline{e}'_n$ and a cross-correlation vector $\underline{\phi}'_{nm} = \overline{\alpha_n \underline{e}'_m}$. The extrapolated correlation matrix is readily shown to be

$$E'_n = \Phi_{n,n-1}\, E_{n-1}\, \Phi^T_{n,n-1} \tag{2.14}$$

As a final comment, note that an estimate of the deviation in the angle to be measured at time t_n may be obtained from the extrapolated estimate of $\delta \hat{\underline{x}}_{n-1}$. We have

$$\delta \hat{A}'_n = \underline{h}^T_n K^T\, \delta \hat{\underline{x}}'_n \tag{2.15}$$

and it is this quantity, compared with the measured deviation $\delta \tilde{A}_n$, which is used in arriving at a revised estimate of $\delta \underline{x}_n$.

2.3 Summary of the Navigation and Guidance Equations

In the navigation and guidance theory presented here, the problem of launch guidance from Earth is ignored. It is assumed that the main propulsion stages are completed at time t_L and that the correlation matrix $E_o = E(t_L)$ is specified initially from a statistical knowledge of injection guidance errors. The initial estimate of position and velocity deviation $\delta \hat{\underline{x}}_o = \delta \hat{\underline{x}}(t_L)$ is zero, since, in the absence of any observation, the best unbiased estimate is that the spacecraft is on course.

The time interval from launch to arrival time t_A at the target point is considered to be subdivided into a number of smaller intervals by the sequence of times t_1, t_2, . . . called "decision points". At each decision point one of three possible courses of action is followed: (1) a single observation is made; (2) a velocity correction is implemented; or (3) no action is taken. A revised

estimate of the deviation vector $\delta \underline{x}(t)$ is made at each such point -- the form of the revision depending, of course, on the nature of the decision. Specifically, as shown in Section 4, the revised estimate at the decision time t_n is one of the following:

$$\delta \underline{\hat{x}}_n = \begin{cases} \delta \hat{x}'_n + a_n^{-1} (E'_n K \underline{h}_n - \underline{\phi}'_{nn}) (\delta \tilde{A}_n - \delta \hat{A}'_n) & \text{(measurement)} \\ (I + J B_n) \delta \underline{\hat{x}}'_n & \text{(correction)} \\ \delta \underline{\hat{x}}'_n & \text{(no action)} \end{cases} \tag{2.16}$$

The scalar coefficient a_n is computed from

$$a_n = \underline{h}_n^T K^T E'_n K \underline{h}_n - 2\underline{h}_n^T K^T \underline{\phi}'_{nn} + \overline{\alpha_n^2} \tag{2.17}$$

The rectangular matrix J has six rows and three columns

$$J = \begin{Vmatrix} 0 \\ I \end{Vmatrix} \tag{2.18}$$

and is just the reverse of the K matrix. The matrix B_n is also rectangular having three rows and six columns and is partitioned as shown

$$B_n = \begin{Vmatrix} C_n^* & -I \end{Vmatrix} \tag{2.19}$$

where C_n^* is one of the fundamental navigation matrices described in Section 3.2.

At each decision point it is also necessary to update the correlation matrix E_n and to compute the correlation vector $\underline{\phi}'_{nn} = \Phi_{n, n-1} \underline{\phi}_{n, n-1}$. Thus

$$E_n = \begin{cases} E'_n - a_n^{-1} (E'_n K \underline{h}_n - \underline{\phi}'_{nn}) (E'_n K \underline{h}_n - \underline{\phi}'_{nn})^T & \text{(measurement)} \\ E'_n + J \overline{\eta_n \eta_n^T} J^T & \text{(correction)} \\ E'_n & \text{(no action)} \end{cases} \tag{2.20}$$

$$\underline{\phi}_{n, n-1} = \overline{\alpha_n \underline{f}_{n-1}} + \sum_{\ell=0}^{n-2} \left\{ \prod_{k=\ell+1}^{n-1} F_k \Phi_{k,k-1} \right\} \overline{\alpha_n \underline{f}_\ell} \tag{2.21}$$

where

$$F_k = \begin{cases} I - a_k^{-1} \left(E_k' \, K \underline{h}_k - \underline{\phi}_{kk}' \right) \underline{h}_k^T \, K^T & \text{(measurement)} \\ I & \text{(correction)} \\ I & \text{(no action)} \end{cases} \tag{2.22}$$

$$\underline{f}_\ell = \begin{cases} a_\ell \, a_\ell^{-1} \left(E_\ell' \, K \underline{h}_\ell - \underline{\phi}_{\ell\ell}' \right) & \text{(measurement)} \\ J \, \underline{\eta}_\ell & \text{(correction)} \\ 0 & \text{(no action)} \end{cases} \tag{2.23}$$

The vector $\underline{\eta}_\ell$ is the difference between the commanded velocity correction and the actual velocity change implemented at time t_ℓ.

The above collection of formulae provides the means of maintaining an up to date estimate of the deviation vector $\delta \hat{\underline{x}}_n$ but, in themselves, do not provide any clue as to what decision should be made at each point. Suggestions for reasonable decision rules are discussed in Section 6.2 and in Appendix B.

It is important to note that the navigation formulae are considerably simplified if the measurement errors are considered uncorrelated. For this special case all the $\underline{\phi}$ correlation vectors are identically zero so that far less computation is required.

3. FUNDAMENTAL NAVIGATION MATRICES

Basic to the solution of the navigation problem is a certain collection of matrices. The objective here is to introduce these matrices, indicate their role in the navigation theory, and show how they may be obtained as solutions of differential equations.

3.1 General Solution of the Linearized Trajectory Equations

Let $\underline{r}_s(t)$ and $\underline{v}_s(t)$ denote the position and velocity vectors of the spacecraft in an inertial coordinate system, and let $\underline{g}(\underline{r}_s, t)$ denote the gravitational acceleration at position $\underline{r}_s$ and time t. Then

$$\frac{d\underline{r}_s}{dt} = \underline{v}_s \quad , \quad \frac{d\underline{v}_s}{dt} = \underline{g}(\underline{r}_s, t) \tag{3.1}$$

are the basic equations of motion of the spaceship except for those brief periods during which propulsion is applied.

Let the vectors $\underline{r}_o(t)$ and $\underline{v}_o(t)$ represent the position and velocity at time t associated with the prescribed *reference trajectory*, and define

$$\delta\underline{r}(t) = \underline{r}_s(t) - \underline{r}_o(t) \quad , \quad \delta\underline{v}(t) = \underline{v}_s(t) - \underline{v}_o(t) \tag{3.2}$$

Then, the deviations $\delta\underline{r}$ and $\delta\underline{v}$ may be approximately related by means of the linearized differential equations:

$$\frac{d(\delta\underline{r})}{dt} = \delta\underline{v} \quad , \quad \frac{d(\delta\underline{v})}{dt} = G(\underline{r}_o, t)\ \delta\underline{r} \tag{3.3}$$

where $G(\underline{r}_o, t)$ is a matrix whose elements are the partial derivatives of the components of $g(\underline{r}_o, t)$ with respect to the components of $\underline{r}_o$.

A particularly useful fundamental set of solutions of Eqs. (3.3) may be developed in the following way. Let t_L and t_A be, respectively, the time of launch and the time of arrival at the target. Then, define the matrices $R(t)$, $R^*(t)$, $V(t)$, $V^*(t)$ as the solutions of the matrix differential equations

$$\begin{aligned} \frac{dR}{dt} &= V \quad , \quad \frac{dR^*}{dt} = V^* \\ \frac{dV}{dt} &= GR \quad , \quad \frac{dV^*}{dt} = GR^* \end{aligned} \tag{3.4}$$

which satisfy the initial conditions

$$R(t_L) = O \quad , \quad R^*(t_A) = O$$
$$V(t_L) = I \quad , \quad V^*(t_A) = I \tag{3.5}$$

Here O and I denote, respectively, the zero and identity matrix. If we now write

$$\delta \underline{r}(t) = R(t)\,\underline{c} + R^*(t)\,\underline{c}^* \tag{3.6}$$

$$\delta \underline{v}(t) = V(t)\,\underline{c} + V^*(t)\,\underline{c}^* \tag{3.7}$$

where $\underline{c}$ and $\underline{c}^*$ are arbitrary constant vectors, it follows that these expressions satisfy the perturbation differential equations (3.3), and contain precisely the required number of unspecified constants to meet any valid set of initial or boundary conditions.

The elements of the R and V matrices represent deviations in position and velocity from the corresponding reference quantities as the result of certain specific deviations in the launch velocity from its reference value. For example, the first columns of these matrices are the vector deviations at time t due to a unit change in the first component of the velocity at time t_L. Corresponding interpretations may be ascribed to the other columns as well. A similar discussion will provide a physical meaning for the elements of R^* and V^*. For this purpose, however, it is convenient to imagine the roles of launch and target points as reversed.

3.2 The Vector Velocity Correction

Associated with the position $\underline{r}_s$ and the time t is the vector velocity required by the spacecraft to travel in free fall from $\underline{r}_s(t)$ to the target point $\underline{r}_o(t_A)$ in the time $t_A - t$. An expression for this velocity vector is readily obtained from Eqs. (3.6) and (3.7). The condition that the vehicle pass through the target point is met by the requirement

$$\delta \underline{r}(t_A) = 0 = R(t_A)\,\underline{c} + R^*(t_A)\,\underline{c}^*$$

Since $R^*(t_A) = O$, it follows that $\underline{c} = 0$. Eliminating $\underline{c}^*$ between Eqs. (3.6) and (3.7) gives for the required velocity deviation* at time t

$$\delta \underline{v}^+(t) = V^*(t)\,R^*(t)^{-1}\,\delta \underline{r}(t) \tag{3.8}$$

*The superscripts - and + are used to distinguish the velocity just prior to correction from the velocity immediately following the correction.

Hence, the required velocity correction $\Delta \underline{v}^*$ is given by

$$\Delta\underline{v}^*(t) = C^*(t)\ \delta\underline{r}(t) - \delta\underline{v}^-(t) \tag{3.9}$$

where the C^* matrix is defined by

$$C^*(t) = V^*(t)\ R^*(t)^{-1} \tag{3.10}$$

The elements of the C^* matrix are deviations in vehicle velocity from the reference values, as required to place the vehicle on a trajectory to the target point, which arise from certain specific deviations in the vehicle position. The interpretation applied to the columns is made in the manner described earlier in connection with the R and V matrices.

If the spacecraft has been in a free-fall status since launch, then, by employing arguments similar to those used in establishing Eq. (3.8), it can be shown that

$$\delta\underline{v}^-(t) = C(t)\ \delta\underline{r}(t) \tag{3.11}$$

where

$$C(t) = V(t)\ R(t)^{-1} \tag{3.12}$$

In this case Eq. (3.9) takes the form

$$\Delta\underline{v}^*(t) = [C^*(t) - C(t)]\ \delta\underline{r}(t) \tag{3.13}$$

Since $\delta\underline{r}(t)$ is different from zero solely as a result of an injection velocity error $\delta\underline{v}(t_L)$, it follows, from the definition of the R matrix, that

$$\Delta\underline{v}^*(t) = -\Lambda(t)\ \delta\underline{v}(t_L) \tag{3.14}$$

Thus, the Λ matrix, defined by

$$\Lambda(t) = V(t) - C^*(t)\, R(t) \tag{3.15}$$

relates a deviation in launch velocity to the velocity impulse required at time t. A starred form of the Λ matrix

$$\Lambda^*(t) = V^*(t) - C(t)\ R^*(t) \tag{3.16}$$

will occur in the subsequent discussions.

3.3 Differential Equation Solutions

The matrices C, C^*, Λ, Λ^* may also be generated directly as solutions of differential equations. However, for C and C^*, a difficulty arises in prescribing appropriate initial conditions. From the initial values of the R and R^* matrices, it follows that $C(t_L)$ and $C^*(t_A)$ are both infinite. The singularities may be avoided by working directly with the differential equation for the inverse matrices C^{-1} and C^{*-1}.

By differentiating the identity

$$C(t)^{-1}\, V(t) = R(t) \tag{3.17}$$

and using Eq. (3.4), the following equation for C^{-1} results

$$\frac{dC^{-1}}{dt} + C^{-1}\, G\, C^{-1} = I \tag{3.18}$$

Similarly, we obtain

$$\frac{dC^{*-1}}{dt} + C^{*-1}\, G\, C^{*-1} = I \tag{3.19}$$

Equations (3.18) and (3.19) may be used to demonstrate an interesting property possessed by C and C^*. It is easy to show that the G matrix is symmetrical. It follows at once that the matrices C and C^* will be symmetrical for all values of t in the interval (t_L, t_A) if they are symmetrical for any particular time. But from Eq. (3.17) and a similar one involving starred matrices, we have

$$C(t_L)^{-1} = 0 \quad , \quad C^*(t_A)^{-1} = 0 \tag{3.20}$$

so that C and C^* are, indeed, symmetrical for t equal to t_L and t_A respectively. Hence C(t) and $C^*(t)$ are symmetrical for all t in the interval from launch to the target point.

In an entirely analogous manner, differential equations may be developed for Λ and Λ^*. By differentiating Eqs. (3.15) and (3.16) and using Eq. (3.4), one readily obtains the equations

$$\frac{d\Lambda}{dt} + C^*\Lambda = 0 \tag{3.21}$$

and

$$\frac{d\Lambda^*}{dt} + C\, \Lambda^* = 0 \tag{3.22}$$

with the initial conditions

$$\Lambda(t_L) = I \quad , \quad \Lambda^*(t_A) = I \tag{3.23}$$

3.4 The State Transition Matrix

Let $\delta \underline{r}_n = \delta \underline{r}(t_n)$ and $\delta \underline{v}_n = \delta \underline{v}(t_n)$ be the deviations in position and velocity at time t_n, and let R_n, V_n, . . . be the corresponding values of the fundamental matrices. The $\underline{c}$ and $\underline{c}^*$ must be obtained as solutions of

$$\delta \underline{r}_n = R_n\, \underline{c} + R_n^*\, \underline{c}^* \tag{3.24}$$

$$\delta \underline{v}_n = V_n\, \underline{c} + V_n^*\, \underline{c}^* \tag{3.25}$$

Multiplying Eq. (3.24) by R_n^{-1}, we obtain for $\underline{c}$

$$\underline{c} = R_n^{-1} (\delta \underline{r}_n - R_n^* \underline{c}^*) \tag{3.26}$$

Then, by substituting this expression into Eq. (3.25) and using Eqs. (3.12) and (3.16), there results

$$\underline{c}^* = - \Lambda_n^{*-1} (C_n \delta \underline{r}_n - \delta \underline{v}_n) \tag{3.27}$$

Finally, from Eq. (3.26) we have

$$\underline{c} = - \Lambda_n^{-1} (C_n^* \delta \underline{r}_n - \delta \underline{v}_n) \tag{3.28}$$

after some simplification. Thus, with $\underline{c}$ and $\underline{c}^*$ determined, the position and velocity deviations at any other time t are given by Eqs. (3.6) and (3.7).

In terms of the six dimensional deviation vector defined by Eq. (2.2), the result may be written in the form

$$\delta \underline{x}(t) = \begin{Vmatrix} R(t) & R^*(t) \\ V(t) & V^*(t) \end{Vmatrix} \begin{Vmatrix} \underline{c} \\ \underline{c}^* \end{Vmatrix} \tag{3.29}$$

Consider now a specific value of $t = t_{n+1}$. Then substituting from Eqs. (3.27) and (3.28) into Eq. (3.29), a relationship between $\delta \underline{x}_{n+1}$ and $\delta \underline{x}_n$ is displayed

$$\delta \underline{x}_{n+1} = \Phi_{n+1,n} \delta \underline{x}_n \tag{3.30}$$

where $\Phi_{n+1,n}$, the six-dimensional state transition matrix, is computed from

$$\Phi_{n+1,n} = \begin{Vmatrix} R_{n+1} & R_{n+1}^* \\ V_{n+1} & V_{n+1}^* \end{Vmatrix} \begin{Vmatrix} -\Lambda_n^{-1} & 0 \\ 0 & -\Lambda_n^{*-1} \end{Vmatrix} \begin{Vmatrix} C_n^* & -I \\ C_n & -I \end{Vmatrix} \tag{3.31}$$

4. DERIVATION OF THE OPTIMUM LINEAR ESTIMATE

As noted in the Introduction, the optimum linear estimate of the deviation vector may be expressed as a recursion formula. Therefore, assume $\delta \hat{\underline{x}}_{n-1}$ and E_{n-1} are known and that a single measurement of the type described in Appendix A is made at time t_n. The observed deviation in the measured quantity A_n is $\delta \tilde{A}_n$, and the best estimate for δA_n, as obtained from the extrapolated estimate of $\delta \hat{\underline{x}}_{n-1}$, is given by Eq. (2.15). Then a linear estimate for the deviation vector $\delta \underline{x}_n$ at time t_n is expressible as a linear combination of the extrapolated estimate of $\delta \underline{x}_{n-1}$ and the difference between the observed and estimated deviations in the measured quantity A_n. Thus

$$\delta \hat{\underline{x}}_n = \delta \hat{\underline{x}}'_n + \underline{w}_n (\delta \tilde{A}_n - \delta \hat{A}'_n) \tag{4.1}$$

where the vector $\underline{w}_n$ is a weighting factor which will be chosen so as to minimize the mean-squared error in the estimate.

For this purpose use Eqs. (2.9), (2.7) and (2.5) to write

$$\begin{aligned} \underline{e}_n(\underline{w}_n) &= \delta \hat{\underline{x}}_n - \delta \underline{x}_n \\ &= \delta \hat{\underline{x}}'_n + \underline{w}_n (\delta A_n + \alpha_n - \delta \hat{A}'_n) - \delta \underline{x}_n \\ &= (I - \underline{w}_n \underline{h}_n^T K^T)(\delta \hat{\underline{x}}'_n - \delta \underline{x}_n) + \underline{w}_n \alpha_n \\ &= (I - \underline{w}_n \underline{h}_n^T K^T) \underline{e}'_n + \underline{w}_n \alpha_n \end{aligned} \tag{4.2}$$

where I is the six-dimensional identity matrix. Then the correlation matrix E_n defined by Eq. (2.11) may be expressed as a function of the weighting vector $\underline{w}_n$ as

$$\begin{aligned} E_n(\underline{w}_n) &= (I - \underline{w}_n \underline{h}_n^T K^T) E'_n (I - K \underline{h}_n \underline{w}_n^T) \\ &+ (I - \underline{w}_n \underline{h}_n^T K^T) \underline{\phi}'_{nn} \underline{w}_n^T \\ &+ \underline{w}_n \phi'^T_{nn} (I - K \underline{h}_n \underline{w}_n^T) + \underline{w}_n \underline{w}_n^T \overline{\alpha_n^2} \end{aligned} \tag{4.3}$$

The mean-squared errors in the estimate of position and velocity deviations $\overline{\epsilon_n^2}$ and $\overline{\delta_n^2}$ are simply the respective traces of the submatrices

$E_n^{(1)}$ and $E_n^{(4)}$. If the six-dimensional weighting vector $\underline{w}_n$ is partitioned into two three-dimensional vectors

$$\underline{w}_n = \left\| \begin{matrix} \underline{w}_n^{(1)} \\ \\ \underline{w}_n^{(2)} \end{matrix} \right\| \tag{4.4}$$

then from Eq. (4.3) it is easy to show that $E_n^{(1)}$ is a function only of $\underline{w}_n^{(1)}$ and $E_n^{(4)}$ is a function only of $\underline{w}_n^{(2)}$. Therefore, for the purposes of the following discussion, it is legitimate formally to treat the mean-squared error in the estimate $\overline{e_n^2(\underline{w}_n)}$ as the trace of the six-dimensional correlation matrix $E_n(\underline{w}_n)$. The subvectors of the optimum weighting vector $\underline{w}_n$ will then each be optimum for the respective estimates of position and velocity deviations.

In order to determine the optimum weighting vector, one may apply the usual technique of the variational calculus. Let $\underline{w}_n$ take on a variation $\delta \underline{w}_n$ and obtain from Eq. (4.3)

$$\begin{aligned} \overline{\delta e_n^2(\underline{w}_n)} = 2 \operatorname{tr} \Big[&- \delta \underline{w}_n \underline{h}_n^T K^T E_n' (I - K \underline{h}_n \underline{w}_n^T) \\ &- \delta \underline{w}_n \underline{h}_n^T K^T \underline{\phi}_{nn}' \underline{w}_n^T + \delta \underline{w}_n \underline{\phi}_{nn}'^T (I - K \underline{h}_n \underline{w}_n^T) + \delta \underline{w}_n \underline{w}_n^T \overline{\alpha_n^2} \Big] \end{aligned} \tag{4.5}$$

If $\overline{\delta e^2(\underline{w}_n)}$ is to vanish for all variations $\delta \underline{w}_n$, then it must follow that

$$a_n \underline{w}_n = E_n' K \underline{h}_n - \underline{\phi}_{nn}' \tag{4.6}$$

where the positive scalar quantity a_n is defined by Eq. (2.17).

It can be readily shown that the $\underline{w}_n$ determined from Eq. (4.6) actually does minimize $\overline{e_n^2(\underline{w}_n)}$. Suppose that the optimum $\underline{w}_n$ is replaced by another weighting factor $\underline{w}_n - \underline{y}_n$. Then from Eqs. (4.3) and (2.17)

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \operatorname{tr} \left[E_n' - 2(\underline{w}_n - \underline{y}_n)(\underline{h}_n^T K^T E_n' - \underline{\phi}_{nn}'^T) + a_n (\underline{w}_n - \underline{y}_n)(\underline{w}_n^T - \underline{y}_n^T) \right] \tag{4.7}$$

and using Eq. (4.6)

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \operatorname{tr} \left[E_n' - a_n (\underline{w}_n - \underline{y}_n)(\underline{w}_n^T + \underline{y}_n^T) \right] \tag{4.8}$$

so that

$$\overline{e_n^2(\underline{w}_n - \underline{y}_n)} = \overline{e_n^2(\underline{w}_n)} + a_n \operatorname{tr} (\underline{y}_n \underline{y}_n^T) \tag{4.9}$$

Thus, the mean-squared error is not decreased by perturbing $\underline{w}_n$ if Eq. (4.6) holds.

Having obtained the optimum weighting vector, the expression for the correlation matrix of the estimate errors E_n given by Eq. (4.3) may be written in a more convenient form. Thus, from the definition of a_n in Eq. (2.17),

there results

$$E_n = E_n' (I - K \underline{h}_n \underline{w}_n^T) - \underline{w}_n h_n^T K^T E_n' + \underline{w}_n \underline{\phi}_{nn}'^T + \underline{\phi}_{nn}' \underline{w}_n^T + a_n \underline{w}_n \underline{w}_n^T \quad (4.10)$$

Substituting from Eq. (4.6), the final expression may be written as

$$E_n = E_n' - a_n^{-1} (E_n' K \underline{h}_n - \underline{\phi}_{nn}') (E_n' K \underline{h}_n - \underline{\phi}_{nn}')^T \quad (4.11)$$

For the case in which the measurement errors are uncorrelated, i.e., $\underline{\phi}'_{nm} = 0$ for all n and m, the estimation procedure is complete. Equations (4.1) and (4.11) in the form

$$\delta \hat{\underline{x}}_n = \delta \hat{\underline{x}}_n' + a_n^{-1} E_n' K \underline{h}_n (\delta \tilde{A}_n - \delta \hat{A}_n') \quad (4.12)$$

$$E_n = E_n' - a_n^{-1} (E_n' K \underline{h}_n)(E_n' K \underline{h}_n)^T$$

then serve as recursive relations to be used in obtaining improved estimates of position and velocity deviations at each of the measurement times $t_1, t_2, \ldots$

If the measurement errors are correlated, the procedure must be expanded to include a method for computing $\underline{\phi}'_{nn}$. For this purpose, a recursion formula may be developed from Eq. (4.2). We have at time t_{m-1}

$$\underline{e}_{m-1} = (I - \underline{w}_{m-1} \underline{h}_{m-1}^T K^T) \underline{e}_{m-1}' + \underline{w}_{m-1} \alpha_{m-1} \quad (4.13)$$

so that

$$\underline{\phi}_{nm}' = \Phi_{m,m-1} (I - \underline{w}_{m-1} \underline{h}_{m-1}^T K^T) \underline{\phi}_{n,m-1}' + \Phi_{m,m-1} \underline{w}_{m-1} \overline{\alpha_n \alpha_{m-1}} \quad (4.14)$$

$$m = 2, 3, \ldots, n$$

Equation (4.14) permits the calculation of $\underline{\phi}'_{nn}$ by successive substitution beginning with m = 2 and noting that

$$\underline{\phi}_{n,1}' = \Phi_{1,0} \overline{\alpha_n \underline{e}_o} \quad (4.15)$$

where $\underline{e}_o = -\delta \underline{x}_o$ is the negative of the deviation vector at time t_2 arising from improper injection into orbit.

The closed form calculation given in Eq. (2.21) is readily derived from Eq. (4.14).

5. STATISTICAL ANALYSIS OF THE GUIDANCE PROCEDURE

From exact knowledge of the six-dimensional deviation vector $\delta \underline{x}_n$ at time t_n, a velocity correction may be calculated which, if implemented, will insure the vehicle's arrival at a fixed point in space at the required time. However, only the estimate $\delta \hat{\underline{x}}_n$ is available. From this, an estimate of the velocity correction vector $\Delta \hat{\underline{v}}_n$ may be determined from

$$\Delta \hat{\underline{v}}_n = B_n \ \delta \hat{\underline{x}}_n \tag{5.1}$$

where B_n is defined by Eq. (2.19). (Refer to the discussion leading to Eq. (3.9).)

The need for a velocity correction arises solely from improper injection into orbit. If the first such correction is executed perfectly, then, of course, no further corrections are required. However, because of imperfect knowledge of position and velocity obtained from navigational measurements, the commanded velocity change will be in error. Furthermore, the actual velocity change experienced will differ from that commanded because of imperfect instrumentation. Therefore, subsequent corrections will be required to remove the effects produced by earlier inaccuracies.

5.1 Correlation Matrix of the Velocity Correction Vector

For notational purposes in this section, it will be convenient to distinguish the times of velocity correction from the other decision times. Therefore, $t_{c,n}$ will be used to denote the time of the n-th corrective maneuver. An arbitrary number of decision points may, of course, fall between $t_{c,n}$ and $t_{c,n+1}$.

Suppose that corrective action is to be taken at the time $t_{c,n}$. Let the commanded velocity change be $\Delta \hat{\underline{v}}_{c,n}$, while the actual velocity change experienced is $\Delta \underline{v}_{c,n}$. Then, with $\underline{\eta}_{c,n}$ denoting the uncertainty, we have

$$\Delta \hat{\underline{v}}_{c,n} = \Delta \underline{v}_{c,n} + \eta_{c,n} \tag{5.2}$$

The actual velocity change may be expressed as

$$\Delta \underline{v}_{c,n} = B_{c,n} \left(\delta \underline{x}'_{c,n} + \underline{e}'_{c,n}\right) - \underline{\eta}_{c,n} \tag{5.3}$$

where $\underline{e}'_{c,n}$ is the error vector as extrapolated from the last observation point. The minus superscript is used to emphasize that the deviation vector corresponds to the time immediately prior to application of the corrective velocity impulse. Similarly, a plus superscript will distinguish a deviation vector immediately subsequent to a velocity correction.

Just as the extrapolated error vector is altered at an observation point, so also will it change at a correction point. Thus

$$\underline{e}_{c,n} = \underline{e}'_{c,n} + \begin{Vmatrix} 0 \\ \underline{\eta}_{c,n} \end{Vmatrix} \tag{5.4}$$

so that Eq. (5.3) becomes

$$\Delta \underline{v}_{c,n} = B_{c,n} (\delta \underline{x}'_{c,n} + \underline{e}_{c,n}) \tag{5.5}$$

We further note that the correlation matrix of the deviation errors must be updated at a correction point following the application of the velocity correction. It follows from Eq. (5.4) that

$$E_{c,n} = E'_{c,n} + \begin{Vmatrix} 0 & 0 \\ 0 & \overline{\underline{\eta}_{c,n} \underline{\eta}^T_{c,n}} \end{Vmatrix} \tag{5.6}$$

A more convenient form of Eq. (5.5) is necessary for subsequent statistical analysis. Indeed, it is possible to express the velocity correction actually applied at time $t_{c,n}$ directly in terms of the error vectors $\underline{e}_{c,n}$ and $\underline{e}_{c,n-1}$. For this purpose, we use the definition of the transition matrix operator to write Eq. (5.5) as

$$\begin{aligned} \Delta \underline{v}_{c,n} &= B_{c,n} (\Phi_{c,n;c,n-1}\, \delta \underline{x}^+_{c,n-1} + \underline{e}_{c,n}) \\ &= B_{c,n}\, \Phi_{c,n;n-1} \left\{ \delta \underline{x}'_{c,n-1} + \begin{Vmatrix} 0 \\ \Delta \underline{v}_{c,n-1} \end{Vmatrix} \right\} + B_{c,n}\, \underline{e}_{c,n} \end{aligned} \tag{5.7}$$

Further simplification is possible because of the identity

$$B_{c,n}\, \Phi_{c,n;c,n-1} = \Lambda_{c,n} \Lambda^{-1}_{c,n-1} B_{c,n-1} \tag{5.8}$$

which is readily established from Eq. (3.31).

Therefore,

$$\Delta \underline{v}_{c,n} = \Lambda_{c,n} \Lambda^{-1}_{c,n-1} (B_{c,n-1}\, \delta \underline{x}'_{c,n-1} - \Delta \underline{v}_{c,n-1}) + B_{c,n}\, \underline{e}_{c,n} \tag{5.9}$$

However, since Eq. (5.5) obtains at the previous correction time $t_{c,n-1}$, then

$$B_{c,n-1}\, \delta \underline{x}'_{c,n-1} - \delta \underline{v}_{c,n-1} = - B_{c,n-1}\, \underline{e}_{c,n-1} \tag{5.10}$$

Hence, finally, we have

$$\Delta \underline{v}_{c,n} = B_{c,n}\, \underline{e}_{c,n} - \Lambda_{c,n} \Lambda^{-1}_{c,n-1} B_{c,n-1} \underline{e}_{c,n-1} \tag{5.11}$$

as the desired expression or, alternately, using Eq. (5.2), we may write

$$\Delta \hat{\underline{v}}_{c,n} = B_{c,n}\, \underline{e}'_{c,n} - \Lambda_{c,n} \Lambda^{-1}_{c,n-1} B_{c,n-1} \underline{e}_{c,n-1} \tag{5.12}$$

The correlation matrix of the estimated velocity correction vector is found by computing the mathematical expectation of the product of $\Delta \hat{\underline{v}}_{c,n}$ and its transpose. Thus

$$\begin{aligned}\overline{\Delta \hat{\underline{v}}_{c,n} \Delta \hat{\underline{v}}^T_{c,n}} = {} & B_{c,n} E'_{c,n} B^{-1}_{c,n} - B_{c,n} \overline{\underline{e}'_{c,n} \underline{e}^T_{c,n-1}} B^T_{c,n-1} \Lambda^{T-1}_{c,n-1} \Lambda^T_{c,n} \\ & - \Lambda_{c,n} \Lambda^{-1}_{c,n-1} B_{c,n-1} \overline{\underline{e}_{c,n-1} \underline{e}'^T_{c,n}} B^T_{c,n} \\ & + \Lambda_{c,n} \Lambda^{-1}_{c,n-1} B_{c,n-1} E_{c,n-1} B^T_{c,n-1} \Lambda^{T-1}_{c,n-1} \Lambda^T_{c,n}\end{aligned} \tag{5.13}$$

expresses this correlation matrix in terms of the correlation matrices of the errors in the deviation vectors at the two consecutive correction points and a cross-correlation matrix $\overline{\underline{e}'_{c,n} \underline{e}^T_{c,n-1}}$.

The cross-correlation matrix appearing in Eq. (5.13) is most easily computed by a recursive operation. In order to facilitate a description of the procedure, let us identify the time t_m with $t_{c,n-1}$ and assume p observation times before the next velocity correction at time $t_{c,n}$. Then, from the definitions of the quantities involved,

$$\overline{\underline{e}'_{m+1} \underline{e}^T_m} = \Phi_{m+1,m} E_m \tag{5.14}$$

At the k-th observation point, the error vector associated with the deviation vector is

$$\underline{e}_{m+k} = (I - \underline{w}_{m+k} \underline{h}^T_{m+k} K^T)\, \underline{e}'_{m+k} + \underline{w}_{m+k}\, \alpha_{m+k} \tag{5.15}$$

according to Eq. (4.2). Therefore, in general, we have

$$\overline{\underline{e}'_{m+k+1} \underline{e}^T_m} = \Phi_{m+k+1,m+k} \left[\left(I - \underline{w}_{m+k} \underline{h}^T_{m+k} K^T \right) \overline{\underline{e}'_{m+k} \underline{e}^T_m} + \underline{w}_{m+k} \underline{\varphi}^T_{m+k,m} \right] \tag{5.16}$$

Now, starting with Eq. (5.14), a succession of cross-correlation matrices is computed from Eq. (5.16) for k= 1, 2, . . . p. The final calculation produces $\overline{\underline{e}'_{m+p+1} \underline{e}^T_m}$ which is identical with the matrix $\overline{\underline{e}'_{c,n} \underline{e}^T_{c,n-1}}$ required for the evaluation of Eq. (5.13).

The mean-squared estimate of the velocity correction is determined as the trace of the matrix $\overline{\Delta \hat{\underline{v}}_{c,n} \Delta \hat{v}^T_{c,n}}$. As a basis for a decision theory, it is important to know something of the precision of the estimate. Clearly, a velocity correction having a large uncertainty should not be commanded if it is possible to improve substantially the estimate by future observations. The uncertainty $\underline{d}_{c,n}$ in the estimate $\Delta \hat{\underline{v}}_{c,n}$ is simply

$$\underline{d}_{c,n} = \Delta\hat{\underline{v}}_{c,n} - B_{c,n}\ \delta\underline{x}_{c,n} = B_{c,n}\ \underline{e}'_{c,n} \tag{5.17}$$

Hence, the mean-squared uncertainty is determined as the trace of the matrix

$$\overline{\underline{d}_{c,n}\underline{d}^T_{c,n}} = B_{c,n}\ E'_{c,n}\ B^T_{c,n} \tag{5.18}$$

5.2 Uncertainty in the Applied Velocity Correction

In order to complete the statistical analysis of the velocity correction, it is necessary to examine more carefully the vector uncertainty $\underline{\eta}$ in the velocity correction. The inaccuracy in establishing a commanded velocity correction $\Delta \hat{\underline{v}}$ is due to errors in both magnitude and orientation. In the following analysis the two sources of error will be assumed independently random with zero means.

Consider a coordinate system in which the estimated velocity correction vector is along one of the coordinate axes. Then if M is the transformation matrix which relates the selected axis system and the original reference system, we may write

$$\Delta\hat{\underline{v}} = \Delta\hat{\underline{v}}\, M \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\| \tag{5.19}$$

Now, define a random variable κ such that

$$\Delta\underline{v} = (1 + \kappa)\Delta\hat{\underline{v}} \tag{5.20}$$

and let γ be the random angle between $\Delta \hat{\underline{v}}$ and $\Delta \underline{v}$ It will be assumed that both κ and γ are small quantities so that powers and products are negligible compared with unity. The actual vector velocity correction is then

$$\Delta\underline{v} = (1 + \kappa)\ \Delta\hat{\underline{v}} M \left\| \begin{matrix} \gamma \cos\beta \\ \gamma \sin\beta \\ 1 \end{matrix} \right\| \tag{5.21}$$

where β is a polar angle defining the rotation of $\Delta \underline{v}$ with respect to $\Delta \hat{\underline{v}}$. Hence, the uncertainty vector $\underline{\eta}$ is expressible as

$$\underline{\eta} = \Delta\hat{\underline{v}} - \Delta\underline{v} = -\Delta\hat{\underline{v}}\ M \left\{ (1 + \kappa)\gamma \left\| \begin{matrix} \cos\beta \\ \sin\beta \\ 0 \end{matrix} \right\| + \kappa \left\| \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \right\| \right\} \tag{5.22}$$

Assume that κ, γ, β are statistically independent random variables with zero means. Further assume that β is uniformly distributed over the interval 0 to 2 . Then one obtains for the correlation matrix of the velocity correction uncertainty

$$\overline{\underline{\eta}\,\underline{\eta}^T} = \overline{\kappa^2}\;\overline{\Delta\hat{\underline{v}}\;\Delta\hat{\underline{v}}^T} + \frac{\overline{\gamma^2}}{2}\;\overline{\Delta\hat{v}^2}\; M \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Vmatrix} M^T \qquad (5.23)$$

$$= \overline{\kappa^2}\;\overline{\Delta\hat{\underline{v}}\;\Delta\hat{\underline{v}}^T} + \frac{\overline{\gamma^2}}{2}\left(\overline{\Delta\hat{\underline{v}}^T\;\Delta\hat{\underline{v}}}\; I - \overline{\Delta\hat{\underline{v}}\;\Delta\hat{\underline{v}}^T}\right)$$

where I is the three-dimensional identity matrix and $\overline{\kappa^2}$ and $\overline{\gamma^2}$ are the mean-squared valued of κ and γ.

5.3 Miss Distance at the Target

Turning now to the problem of guidance accuracy, the determination of the position deviation vector at the nominal time of arrival at the target is made by extrapolating the deviation vector from the point of the final velocity correction. Thus, if t_N is the time of the last correction and $\delta \underline{x}_A$ is the deviation vector at the time of arrival t_A, then

$$\delta \underline{x}_A = \Phi_{A,N}\; \delta \underline{x}_N^+ \qquad (5.24)$$

But from Eq. (3.31) and the terminal conditions for the navigation matrices, we have

$$\Phi_{A,N} = \begin{Vmatrix} -R_A\,\Lambda_N^{-1} & 0 \\ -V_A\,\Lambda_N^{-1} & -\Lambda_N^{*-1} \end{Vmatrix} \begin{Vmatrix} C_N^* & -I \\ C_N & -I \end{Vmatrix} \qquad (5.25)$$

Hence, the position deviation vector at the target $\delta \underline{r}_A$ may be written as

$$\delta \underline{r}_A = -R_A\,\Lambda_N^{-1}\,B_N\;\delta \underline{x}_N^+ \qquad (5.26)$$

with a similar expression obtainable for the velocity deviation at time t_A.

The target position error may be written ultimately in terms of the error vector $\underline{e}_N$ according to the following self-evident steps

$$
\begin{aligned}
\delta \underline{r}_A &= - R_A \Lambda_N^{-1} B_N \left(\delta \bar{\underline{x}}_N + J \Delta \underline{y}_N\right) \\
&= - R_A \Lambda_N^{-1} \left(B_N \delta \bar{\underline{x}}_N - \Delta \underline{y}_N\right) \\
&= R_A \Lambda_N^{-1} \left(B_N \underline{e}'_N - \underline{\eta}_N\right) \\
&= R_A \Lambda_N^{-1} B_N \underline{e}_N
\end{aligned}
\tag{5.27}
$$

The mean square position error at the target is then computed as the trace of the matrix $\overline{\delta \underline{r}_A \; \delta \underline{r}_A^T}$.

6. APPLICATION TO TRANS-LUNAR NAVIGATION

6.1 Decision Rules

As a necessary step in the application of the navigation and guidance scheme formulated in this paper, certain rules must be adopted concerning the course of action to be taken at each of the "decision points" described in Section 2.3. The number and frequency of observations must be controlled in some manner -- ideally by a decision rule which is realistically compatible with both the mission objectives and the capabilities of the measuring device. If an observation is to be made, a decision is required regarding the type of measurement and the celestial objects to be used. Periodic velocity corrections must be applied and the number of impulses and times of occurrence must be decided.

Once the decision rules have been specified, it is necessary to test their effectiveness according to some measure of performance. A typical objective is to minimize the miss distance at the target. However, a reduction in miss distance usually implies an increase in either the required number of measurements or a greater expenditure of corrective propulsion or both. In the face of these conflicting objectives, compromises are clearly necessary and statistical simulation provides a means of arriving at an acceptable balance.

In the interest of minimizing the number of simulator runs, Monte Carlo techniques should be avoided if possible. Fortunately, it is unnecessary to generate the true spacecraft trajectory, as would be required for Monte Carlo simulation, in order to analyze the effects of a particular set of decision rules. The reader may readily verify that Eq. (2.16), which defines the estimate $\delta \hat{\underline{x}}_n$ and depends on actual measurement data, is never involved in any of the statistical calculations.

A specific example of a set of decision rules to be applied at each decision point is as follows:

1. The estimated mean-squared velocity correction $\overline{\Delta \hat{v}_n^2}$ and the mean-squared uncertainty $\overline{d_n^2}$ associated with the estimate are computed from Eqs. (5.13) and (5.18). If the ratio

$$R_v = \sqrt{\overline{d_n^2} / \Delta \overline{\hat{v}_n^2}} \tag{6.1}$$

is less than a specified amount $R_{v(min)}$, a velocity correction is made at time t_n. In any case, the correction is made, regardless of the value of R_v, if the root- mean-squared velocity correction exceeds a fixed amount Δv_{max}.

2. If neither of the criteria is met which would call for initiation of a velocity correction, the desirability of making an observation is examined. For this purpose, an abbreviated star catalog is postulated together with selected planets. Each star and planet combination is analyzed to determine the effect of each such measurement on the potential reduction in the miss distance at the target. More specifically, at time t_n the mean-square miss distance, which would result if no further corrective action were taken, is calculated from Eq. (5.27) . Now, if any measurement were made and were followed immediately by a velocity correction, a calculable mean-square reduction in the target error would result. The particular star-planet combination producing the greatest mean-square reduction in target error is then defined as the best potential measurement.

Now let $\overline{\delta r_A^{2+}}$ and $\overline{\delta r_A^{2-}}$ be the respective mean-square miss distances which would result with and without the best possible observation. Then, if the ratio

$$R_p = \sqrt{\frac{\overline{\delta r_A^{2-}} - \overline{\delta r_A^{2+}}}{\overline{\delta r_A^{2-}}}} \tag{6.2}$$

is greater than a specified value $R_{p(max)}$, the best potential measurement is made at time t_n. In other words, for a measurement to be made, a significant reduction in the potential miss distance must result. If, on the other hand, the above criterion is not met, no action is taken at the decision point t_n.

6.2 A Numerical Example

The particular set of decision rules formulated in the previous section were applied to the problem of navigating along a trans-lunar trajectory. For simplicity, possible cross-correlation between measurement errors was ignored. Furthermore, only the Earth and the Moon, together with the ten brightest stars, were considered for potential measurements.

For the specific trajectory chosen for illustration, the date and time of orbital injection was Julian Day 2440042.5 at the twelfth hour. The target point was selected approximately 11, 000 miles beyond the moon with the closest point of approach some 3, 000 miles from the lunar surface. The nominal time of flight from injection was 84 hours.

The correlation matrix of injection errors E_o was assumed to be

$$E_o = \begin{Vmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{400}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{400}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{400}{3} \end{Vmatrix}$$

which corresponds to assuming zero mean-squared position error and an isotropic velocity error whose root-mean-square value is 20 miles per hour.

At each decision point, twenty potential measurements were examined and evaluated according to the decision criterion. If the best potential measurement was actually made, then the second best measurement was also evaluated using the same criterion. In this manner, the possibility of two observations at each decision point was allowed.

For simplicity, only star elevations above an illuminated horizon of either the Earth or Moon were considered. Certain practical constraints were imposed so that physically unrealizable measurements were screened out. For example, no measurement could be made if the line of sight to either star or planet edge were closer than fifteen degrees from the direction to the Sun. Furthermore, the relative orientation of the Earth and Moon were taken fully into account; e. g., if the illuminated face of the Moon formed the background of the edge of the Earth from which a star elevation was to be reckoned, that particular measurement would not be made.

The optical measuring device used for the observations was assumed to be unbiased with a random error whose variance was

$$\sigma^2 = (0.00005)^2 + \left(\frac{2}{r_{SP}}\right)^2 \text{ radians}$$

where r_{SP} is the distance in miles from the spacecraft to the planet edge. In this manner it was possible to account for the larger uncertainty in defining the horizon which would exist when the spacecraft is close to the planet. At large distances the rms error is approximately 0.05 milliradians.

Finally, in this example calculation, the error in applying a velocity correction was assumed to be isotropic and proportional to the commanded correction. Specifically, the relation

$$\overline{\eta^2_{c,n}} = 0.0001 \ \overline{\Delta\hat{v}^2_{c,n}}$$

was adopted so that the rms error would be one percent of the rms correction.

Preliminary results of an analysis of this sample trajectory are summarized in the accompanying tables. A number of simulated guidance flights were made for which the strategy parameters R_v and R_p had various assigned values. Certain pertinent navigation data are recorded in the tables as functions of these strategy parameters.

In Table 1 the navigation data is given as a function of the miss distance reduction ratio R_p for a value of the velocity correction uncertainty ratio $R_v = 0.3$. As one requires each measurement to have a proportionately greater significance in the reduction of the potential target error, the total number of measurements is considerably lessened. The extremes are 126 and 22 measurements. The number of corrective velocity impulses remains about the same but the total of the velocity corrections applied increases by fifty percent. The miss distance at the target varies between one and three miles while the uncertainty in the knowledge of the vehicle velocity at the target falls between one-quarter and one-half mile per hour.

The navigation data recorded in Table 2 is a function of the velocity correction uncertainty ratio with R_p fixed at 0.4. The total of 77 measurements did not vary with R_v but the number of velocity corrections decreased from 9 to 4 as R_v was decreased from 0.4 to 0.1. The final miss distance and uncertainty in velocity were not greatly effected and the total of the velocity corrections varied between approximately 65 and 80 miles per hour. Also recorded

in the table is the time from injection to the application of the first velocity correction. As the parameter R_V was decreased, this time increased from two to five hours.

Table 1. Navigation data as a function of miss distance reduction ratio.

Velocity Correction Uncertainty Ratio = 0.3

Miss Distance Reduction Ratio	Number of Measurements	Number of Velocity Corrections	Total Velocity Correction (mph)	Final Miss Distance (miles)	Final Velocity Uncertainty (mph)
0.3	126	7	55.5	1.15	0.23
0.4	77	7	71.1	1.41	0.28
0.5	50	7	73.1	1.90	0.39
0.6	33	6	79.7	2.25	0.45
0.7	22	6	84.7	3.19	0.57

Table 2. Navigation data as a function of velocity correction uncertainty ratio.

Miss Distance Reduction Ratio = 0.4 No. of Measurements = 77

Velocity Correction Uncertainty Ratio	Time of First Velocity Correction	Number of Velocity Corrections	Total Velocity Correction (mph)	Final Miss Distance (miles)	Final Velocity Uncertainty (mph)
0.40	2.0	9	70.2	1.41	0.29
0.30	2.0	7	71.1	1.41	0.28
0.25	2.4	6	70.6	1.45	0.29
0.20	2.6	6	65.9	1.37	0.28
0.15	3.4	5	74.7	1.47	0.30
0.10	5.0	4	79.1	1.52	0.31

In order to evaluate the effect on the navigation data of a variation in the time of year, a number of pseudo-trajectories were generated by the simple device of rotating the direction of the Sun as viewed from the Earth. The trajectory was considered to be unchanged by this process -- the assumption being quite adequate for the purpose of this preliminary analysis. One set of values for R_V and R_p was selected and the Sun direction was altered in sixty degree steps. In this manner different illuminated portions of the Earth and

Mars were visible to the spacecraft resulting, thereby, in different measurements. As seen in Table 3, the total number of measurements varied by twenty percent while the total of the velocity corrections had a variation between 67 and 87 miles per hour. The effects on miss distance and velocity uncertainty were not substantial.

Table 3. Navigation data for pseudo-trajectories as a function of sun direction rotation.

Miss Distance Reduction Ratio = { 0.5 Time = Start to 10 hours; 0.4 Time = 10 hours to 50 hours; 0.5 Time = 50 hours to end }

Velocity Correction Uncertainty Ratio = 0.3

Sun Direction Rotation (degrees)	Number of Measurements	Number of Velocity Corrections	Total Velocity Correction (mph)	Final Miss Distance (miles)	Final Velocity Uncertainty (mph)
0	57	7	73.5	1.70	0.36
60	59	7	68.4	1.82	0.39
120	53	6	67.2	1.75	0.33
180	53	7	75.8	2.06	0.54
240	55	7	75.5	2.11	0.55
300	49	6	86.9	2.21	0.54

Finally, in Table 4, a complete history of one of the guided translunar flights is recorded. With $R_p = 0.7$ and $R_v = 0.3$, the time and nature of each observation and velocity correction is listed. It is curious to note the frequency with which a particular star and planet combination is repeated in the record of observations.

In conclusion, it should be emphasized that only fragmentary results have been obtained by utilizing this navigational scheme. Therefore, too great an importance should not be attached to the numerical data presented in this section. At this point, only the ideas are significant, and one must await a more thorough investigation before definitive conclusions can be drawn.

Table 4. Example of translunar navigation.

Miss Distance Reduction Ratio = 0.7
Velocity Correction Uncertainty Ratio = 0.3

TIME (hours)	OBSERVATION		VELOCITY CORRECTION (mph)
0.70	Earth	Procyon	
1.20	Earth	Procyon	
1.80	Earth	Procyon	
2.00			37.05
2.60	Earth	Procyon	
3.80	Earth	Procyon	
6.00	Earth	Procyon	
6.00	Moon	Capella	
6.80			17.14
10.00	Earth	Procyon	
11.00	Moon	Capella	
15.00	Moon	Procyon	
18.00	Moon	Procyon	
20.00			6.57
22.00	Moon	Procyon	
28.00	Moon	Procyon	
34.00	Moon	Procyon	
42.00	Moon	Capella	
42.00	Earth	Arcturus	
45.00			3.44
51.00	Moon	Capella	
65.00	Moon	Capella	
76.00	Moon	Rigel	
80.20	Moon	Arcturus	
80.40			8.31
81.20	Moon	Capella	
82.50	Moon	Capella	
			12.19

APPENDIX A

NAVIGATIONAL MEASUREMENTS

The mathematical processes are considered here in some detail for determining spacecraft position by means of both celestial observation and ground based radar measurements. It is assumed throughout the analysis that approximations to spacecraft position and velocity are already known so that perturbation techniques may be employed.

Secondary effects arising from the finite speed of light, the finite distance or stars, etc. are ignored in this analysis. Such effects may be lumped together for a particular reference point on the trajectory as a modification to the stored data which represent reference values for the quantities to be measured at that point.

For simplicity in the present analysis, it will be assumed that the spacecraft clock is perfect so that all measurements are made at known instants of time. Methods of including clock errors in the computation are discussed thoroughly in reference 2.

As indicated in Section 2.1 each measurement establishes a component of spacecraft position along some direction in space. If A is the quantity to be measured and δA is the difference between the true and the reference values, then it will be shown that the relation between δA and the deviation in spacecraft position $\delta \underline{r}$ is

$$\delta A = \underline{h}^T \delta \underline{r} \tag{A.1}$$

regardless of the type of measurement. Thus, the $\underline{h}$ vector alone will characterize the kind of measurement.

Sun-Planet Measurement

The first type of measurement to be considered is that of the angle from the Sun to a planet. By passing to the limit of infinite distance from one or the other of these bodies, corresponding relations for the Sun-star or planet-star type of measurement may be obtained.

Let S_o and P_o be, respectively, the reference positions of the spacecraft and a planet at the time of the measurement. Let $\underline{r}_s$ be the vector from the Sun to S_o and $\underline{z}$ the vector from S_o to P_o. With A denoting the angle from the Sun line to the planet line, we have

$$\cos A = -(\underline{r} \cdot \underline{z})/rz \tag{A.2}$$

where r and z denote magnitudes of the respective vectors $\underline{r}$ and $\underline{z}$. Treating all changes as first-order differentials, it can be shown that

$$\delta A = \left(\frac{\underline{m} - (\underline{n} \cdot \underline{m})\,\underline{n}}{r \sin A} + \frac{\underline{n} - (\underline{n} \cdot \underline{m})\,\underline{m}}{z \sin A} \right) \cdot \delta \underline{r} \tag{A.3}$$

For details the reader is referred to reference 2. Here $\underline{m}$ and $\underline{n}$ are, respectively, the unit vectors from S_o toward the Sun and toward P_o. The two individual vector coefficients of $\delta\underline{r}$ in Eq. (A.3) are vectors in the plane of the measurement and normal, respectively, to the lines-of-sight to the Sun and to the planet.

Planet Diameter Measurement

If D is the actual diameter of a planet, the apparent angular diameter A is found from

$$\sin(A/2) = D/2z \tag{A.4}$$

Again taking differentials as before, one can show that

$$\delta A = \frac{D\underline{m} \cdot \delta \underline{r}}{z^2 \cos(A/2)} \tag{A.5}$$

Star Occultations

The next type of measurement to be considered is that of noting the time at which a star is occulted by a planet. Let $\underline{z}$ be the vector from S_o to P_o, $\underline{r}$ the vector from the Sun to S_o and $\underline{n}$ a unit vector in the direction of the star to be occulted. With γ denoting the angle from the star line to the planet line as shown in Fig. A-1, we have, at the nominal instant of occultation,

$$\underline{n} \cdot \underline{z} = z \cos \gamma \tag{A.6}$$

Treating changes as first order differentials we obtain

$$\begin{aligned} \underline{n} \cdot \delta \underline{z} &= \cos\gamma\, \delta z - z \sin\gamma\, \delta\gamma \\ &= \cos\gamma\, \underline{m} \cdot \delta \underline{z} - z \sin\gamma\, \delta\gamma \end{aligned} \tag{A.7}$$

where $\underline{m}$ is a unit vector from S_o toward P_o.

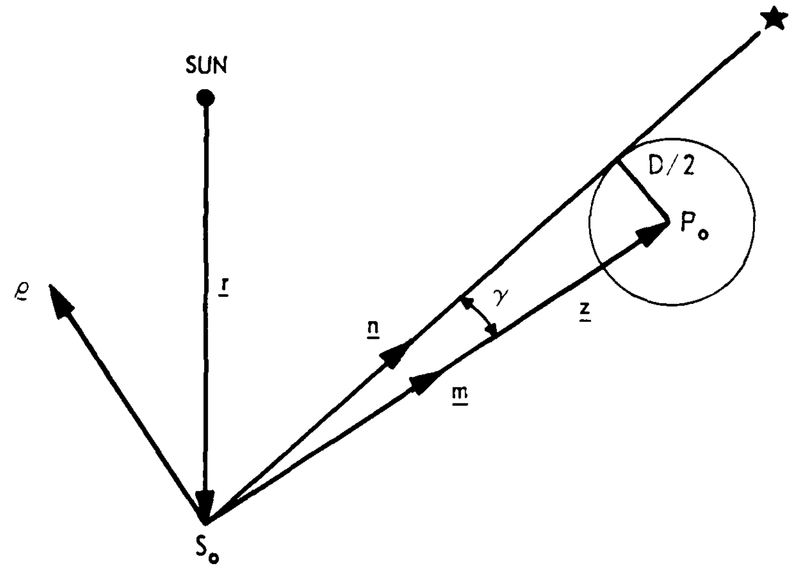

Fig. A-1. Measurement of time of a star occultation.

The angle deviation $\delta\gamma$ is computed from a first order differential of $2z \sin\gamma = D$. There results

$$\delta\gamma = - D\underline{m} \cdot \delta\underline{z}/2z^2 \cos\gamma \qquad (A.8)$$

Furthermore, if $\underline{v}_p$ and $\underline{v}_s$ are the respective velocity vectors of the planet and the spacecraft and if $\delta\tau$ is the difference between the observed and the reference occultation times, we have

$$\begin{aligned} \delta\underline{z} &= \underline{v}_p\, \delta\tau - (\delta\underline{r} + \underline{v}_s\, \delta\tau) \qquad (A.9) \\ &= - \delta\underline{r} - \underline{v}_r\, \delta\tau \end{aligned}$$

where $\underline{v}_r$ is the velocity of the spacecraft relative to the planet. Then by combining Eqs. (A.7), (A.8) and (A.9) we have finally

$$\delta\tau = - \frac{1}{(\underline{\rho} - \tan\gamma\, \underline{m}) \cdot \underline{v}_r} (\underline{\rho} - \tan\gamma\, \underline{m}) \cdot \delta\underline{r} \qquad (A.10)$$

where $\underline{\rho}$ is a unit vector perpendicular to $\underline{m}$ and lying in the plane determined by the lines-of-sight to the planet and the star.

Star Elevation Measurement

Consider next the measurement of the angle between the lines-of-sight to a star and the edge of a planet disc. From Fig. A-2 we have

$$\underline{n} \cdot \underline{z} = z \cos(A + \gamma) \qquad (A.11)$$

where A is the angle to be measured. Again taking total differentials and noting that $\delta \underline{r} = - \delta \underline{z}$, we obtain

$$\frac{1}{z}\underline{\rho} \cdot \delta \underline{r} = \delta A + \delta \gamma \tag{A.12}$$

$$= \delta A + D\underline{m} \cdot \delta \underline{r}/2z^2 \cos \gamma$$

$$= \delta A + \tan \gamma \, \underline{m} \cdot \delta \underline{r}/z$$

or finally

$$\delta A = \frac{1}{z}(\underline{\rho} - \tan \gamma \, \underline{m}) \cdot \delta \underline{r} \tag{A.13}$$

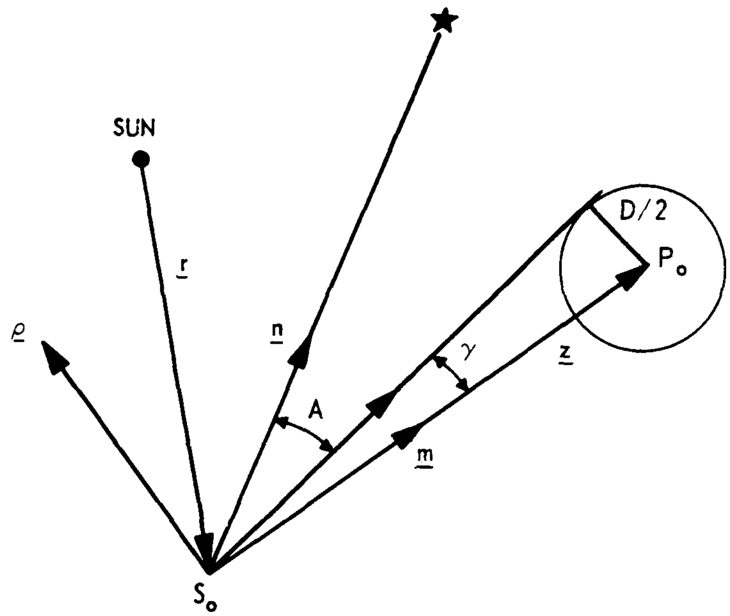

Fig. A-2. Measurement of star elevation angle.

Landmark Measurement

For the measurement of the angle between a landmark on a planet surface and a star, let $\underline{\rho}$ be a unit vector perpendicular to the line-of-sight to the landmark and in the plane of the measurement. Then if $\underline{p}$ is the vector position of the landmark relative to the center of the planet, we have

$$\delta A = \frac{\underline{\rho} \cdot \delta \underline{r}}{|\underline{z} + \underline{\rho}|} \tag{A.14}$$

Radar Range, Azimuth, and Elevation Measurements

Assume the radar site to be the origin of the coordinate system although other origins could equally well be used. Let a cartesian coordinate system be chosen such that the z axis is radially out from the center of the Earth through the radar site; the x axis is positive in the direction from which radar azimuths are to be measured; the y axis completes the coordinate system. Then, we may write

$$\underline{r} = r \begin{Vmatrix} \cos\beta\cos\theta \\ \cos\beta\sin\theta \\ \sin\beta \end{Vmatrix} \tag{A.15}$$

where r, θ, β are, respectively, the range, azimuth, and elevation of the vehicle from the radar site. Taking differentials separately for each of the three variables gives

$$\delta\underline{r} = \begin{Vmatrix} \cos\beta\cos\theta \\ \cos\beta\sin\theta \\ \sin\beta \end{Vmatrix} \delta r \tag{A.16}$$

$$\delta\underline{r} = r \begin{Vmatrix} -\sin\beta\cos\theta \\ -\sin\beta\sin\theta \\ \cos\beta \end{Vmatrix} \delta\beta \tag{A.17}$$

$$\delta\underline{r} = r \begin{Vmatrix} -\cos\beta\sin\theta \\ \cos\beta\cos\theta \\ 0 \end{Vmatrix} \delta\theta \tag{A.18}$$

Then, by expressing each of these relations in the form of Eq. (A.1), we obtain

$$\delta r = \begin{Vmatrix} \cos\beta\cos\theta & \cos\beta\sin\theta & \sin\beta \end{Vmatrix} \delta\underline{r} \tag{A.19}$$

$$\delta\beta = \frac{1}{r} \begin{Vmatrix} -\sin\beta\cos\theta & -\sin\beta\sin\theta & \cos\beta \end{Vmatrix} \delta\underline{r} \tag{A.20}$$

$$\delta\theta = \frac{1}{r\cos\beta} \begin{Vmatrix} -\sin\theta & \cos\theta & 0 \end{Vmatrix} \delta\underline{r} \tag{A.21}$$

APPENDIX B

OPTIMUM SELECTION OF NAVIGATION MEASUREMENTS

In the main body of this paper a method of processing measurement data in an optimum linear manner has been developed. The purpose of this appendix is to treat the associated problem of selecting those measurements which are, in some sense, most effective. For example, the requirement might be to select the measurement to be made at time t_n in order to get the maximum reduction in mean-squared positional or velocity uncertainty at time t_n. Of perhaps greater significance would be the requirement of selecting the measurement which minimizes the uncertainty in any linear combination of position and velocity deviations. Specifically, one might select the measurement which minimizes the uncertainty in the required velocity correction. As a further example, one might wish to select that measurement which, if followed immediately by a velocity correction, would result in the smallest position error at the target.

Consider first the simplest case, i.e., minimizing the mean-squared positional uncertainty at time t_n. From Eq. (4.12) the mean-squared positional uncertainty is expressible as

$$\overline{\epsilon_n^2} = \mathrm{tr}\left(E_n^{(1)'}\right) - \frac{\underline{h}_n^T E_n^{(1)'} E_n^{(1)'} \underline{h}_n}{\underline{h}_n^T E_n^{(1)'} \underline{h}_n + \overline{\alpha_n^2}} \tag{B.1}$$

assuming the measurement errors to be uncorrelated. In the absence of any measurement error ($\alpha_n^2 = 0$) , the problem of minimizing either mean-squared error is equivalent to finding a direction for the $\underline{h}_n$ vector which maximizes the ratio of two quadratic forms. For the case of the mean-squared positional error, the geometrical interpretation is clear. Since the principal directions of $E_n^{(1)'}$ and $E_n^{(1)'} E_n^{(1)'}$ are the same, the optimal direction for $\underline{h}_n$ coincides with the major principal direction of $E_n^{(1)}$.

The problem of minimizing the mean-squared velocity uncertainty at time t_n by proper choice of the $\underline{h}_n$ vector is not as easily solved or interpreted. Again, from Eq. (4.12) the mean-squared velocity uncertainty may be written as

$$\overline{\delta_n^2} = tr\,(E_n^{(4)'}) - \frac{\underline{h}_n^T\, E_n^{(2)'}\, E_n^{(3)'}\, \underline{h}_n}{\underline{h}_n^T\, E_n^{(1)'}\, \underline{h}_n + \overline{\alpha_n^2}} \tag{B.2}$$

Denote by p and q the two quadratic forms

$$p = \underline{h}_n^T\, E_n^{(2)'}\, E_n^{(3)'}\, \underline{h}_n \quad , \quad q = \underline{h}_n^T\, E_n^{(1)'}\, \underline{h}_n \tag{B.3}$$

From the theory of quadratic forms there exists an orthogonal transformation which will reduce q to a diagonal form. Thus

$$\underline{h}_n = Q\,\underline{d} \tag{B.4}$$

gives

$$q = \underline{d}^T\, Q^T\, E_n^{(1)'}\, Q\,\underline{d} = \mu_1\, d_1^2 + \mu_2\, d_2^2 + \mu_3\, d_3^2 \tag{B.5}$$

where μ_1, μ_2, μ_3 are the characteristic roots of the matrix $E_n^{(1)'}$ and the columns of the Q matrix are the associated characteristic unit vectors. Since $E_n^{(1)'}$ is a positive definite matrix, the characteristic roots are positive and a further transformation

$$\underline{f} = D\,\underline{d} \tag{B.6}$$

gives

$$q = \underline{f}^T\, \underline{f} = f_1^2 + f_2^2 + f_3^2 \tag{B.7}$$

where D is a diagonal matrix whose diagonal elements are $\sqrt{\mu_1}$, $\sqrt{\mu_2}$, $\sqrt{\mu_3}$.

The same transformation from $\underline{h}_n$ to $\underline{f}$ applied to the quadratic form p produces

$$p = \underline{f}^T\, D^{-1}\, Q^T\, E_n^{(2)'}\, E_n^{(3)'}\, Q\, D^{-1}\, \underline{f} \tag{B.8}$$

One final transformation applied to $\underline{f}$ will reduce Eq. (B.8) to a diagonal form thus

$$\underline{f} = S\,\underline{m} \tag{B.9}$$

results in

$$p = \lambda_1\, m_1^2 + \lambda_2\, m_2^2 + \lambda_3\, m_3^2 \tag{B.10}$$

where the columns of the S matrix are the characteristic unit vectors of the matrix $D^{-1} Q^T E_n^{(2)\prime} E_n^{(3)\prime} Q D^{-1}$ and λ_1, λ_2, λ_3, the corresponding characteristic roots. The same transformation (B.9) applied to (B.7) gives

$$q = \underline{m}^T S^T S \underline{m} = m_1^2 + m_2^2 + m_3^2 \tag{B.11}$$

since S is an orthogonal matrix.

In summary, then, the transformation

$$\underline{h}_n = Q D^{-1} S \underline{m} \tag{B.12}$$

produces for the ratio of the two quadratic forms

$$\frac{p}{q} = \frac{\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2}{m_1^2 + m_2^2 + m_3^2} \tag{B.13}$$

Furthermore, if the matrix $E_n^{(2)\prime}$ is nonsingular, the product $E_n^{(2)\prime} E_n^{(3)\prime} = E_n^{(2)\prime} E_n^{(2)\prime T}$ is positive definite and it would then follow that λ_1, λ_2, λ_3 are all real and positive.

The problem of maximizing the ratio p/q is now readily solved. Since no measurement error is assumed, one cannot hope to determine more than the direction for the optimum $\underline{h}_n$ or, equivalently, the optimum $\underline{m}$. Therefore, it may be assumed that $\underline{m}$ is a unit vector. Let

$$\lambda_k = \max (\lambda_1, \lambda_2, \lambda_3) \tag{B.14}$$

Then the optimum value of $\underline{m}$ is

$$m_j = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases} \tag{B.15}$$

The same technique can be used to select that direction for $\underline{h}_n$ which minimizes the uncertainty in any linear combination of position and velocity deviations. Specifically, consider the selection of that measurement which minimizes the uncertainty in the velocity correction which would be required immediately following the measurement.

The correlation matrix of the velocity correction uncertainty is

$$\overline{\underline{d}_n \underline{d}_n^T} = B_n E_n B_n^T \tag{B.16}$$

and the mean-squared uncertainty may be expressed as

$$\overline{d_n^2} = \mathrm{tr}\,(B_n E_n' B_n^T) - \frac{\underline{h}_n^T W \underline{h}_n}{\underline{h}_n^T E_n^{(1)\prime} \underline{h}_n + \overline{\alpha_n^2}} \tag{B.17}$$

Here W is a symmetric matrix defined by

$$W = \left\| E_n^{(1)'} \; E_n^{(2)'} \right\| \; B_n^T \; B_n \left\| \begin{matrix} E_n^{(1)'} \\ \\ E_n^{(2)'T} \end{matrix} \right\| \tag{B.18}$$

so that if $\left\| E_n^{(1)'} \quad E_n^{(2)'} \right\| \; B_n^T$ is nonsingular, the matrix W will be positive definite. Under any circumstances, if the identification

$$E_n^{(2)'} \sim \left\| E_n^{(1)'} \quad E_n^{(2)'} \right\| \; B_n^T$$

is made, then the exact same procedure may be used to select the optimum direction for the $\underline{h}_n$ vector as was used previously to minimize the mean-squared velocity uncertainty.

In all cases of practical interest the determination of the optimum direction for the $\underline{h}_n$ vector must be made subject to certain constraints. For example, one might wish to select the "best" star to be used in measuring the angle between the line of sight to the center of a planet disc and the line of sight to the star. For such a measurement the $\underline{h}_n$ vector is required to be perpendicular to the line of sight to the planet. If $\underline{z}_n$ is the position vector of the planet from the space vehicle, then we must have

$$\underline{h}_n^T \, \underline{z}_n = 0 \tag{B.19}$$

Applying the transformation defined in Eq. (B.12) gives

$$\underline{m}^T \, S^T \, D^{-1} \, Q^T \, \underline{z}_n = 0 \tag{B.20}$$

Let $\underline{p}$ be a unit vector in the direction of $S^T D^{-1} Q^T \underline{z}_n$. Then the problem of selecting the optimum direction for $\underline{h}_n$ or, equivalently, for $\underline{m}$ is to maximize

$$\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 m_3^2$$

subject to the conditions of constraint

$$\underline{m}^T \, \underline{p} = 0 \text{ and } \underline{m}^T \, \underline{m} = 1 \tag{B.21}$$

In terms of the Lagrange multipliers ρ and σ, this is equivalent to the problem of obtaining a free maximum for

$$\sum_{j=1}^{3} \lambda_j m_j^2 - 2\rho \sum_{j=1}^{3} p_j m_j - \sigma \left\{ \sum_{j=1}^{3} m_j^2 - 1 \right\}$$

Setting the partial derivatives with respect to each of the m_j's equal to zero, we have

$$m_j = \frac{\rho p_j}{\lambda_j - \sigma} \qquad j = 1, 2, 3 \tag{B.22}$$

where ρ and σ are to be determined from the requirements of Eq. (B-21).

The condition that $\underline{m}$ be orthogonal to $\underline{p}$ leads to a quadratic equation for σ,

$$\sigma^2 - \left[p_1^2 (\lambda_2 + \lambda_3) + p_2^2 (\lambda_1 + \lambda_3) + p_3^2 (\lambda_1 + \lambda_2)\right] \sigma + p_1^2 \lambda_2 \lambda_3 + p_2^2 \lambda_1 \lambda_3 + p_3^2 \lambda_1 \lambda_2 = 0 \tag{B.23}$$

If the λ's are ordered $\lambda_1 < \lambda_2 < \lambda_3$, then the two roots σ_1 and σ_2 will be such that $\lambda_1 < \sigma_1 < \lambda_2 < \sigma_2 < \lambda_3$. One of these roots provides the maximum while the other gives the minimum. The other Lagrange multiplier ρ is determined so that $\underline{m}$ will be a unit vector.

With the optimum vector $\underline{m}$ selected, the corresponding value for $\underline{h}_n$ is found from Eq. (B.12).

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